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CLASSICAL MECHANICS OF AUTOCOMPOSITE PARTICLES

ABSTRACT. A class of theories of fields on $(n+1)$-dimensional space-time valued on a compact n-manifold admit classical localized solutions, which are interpreted as systems of autocomposite particles. A covariant method for extracting the classical mechanics of these autocomposite particles from the underlying field theory is presented.

1. Introduction

The term 'autocomposite particle' was coined by Bernard Jouvet in 1967 [1]. Its meaning is best explained on an example. Consider the model quantum field theory defined by the unrenormalized Lagrangian density

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu \varphi_0 \partial^\mu \varphi_0 - \mu_0^2 \varphi_0^2) + \tfrac{1}{2}(\partial_\mu \psi_0 \partial^\mu \psi_0 - m_0^2 \psi_0^2) + \tfrac{1}{2} g_0 \varphi_0 \psi_0^2. \tag{1.1}$$

In the limit $\mu_0^2 \to \infty, g_0/\mu_0^2$ finite (which corresponds, in particular, to $Z_3 = 0$), the Euler equation for the field $\varphi_0(x)$ reduces to the constraint

$$\varphi_0 = \frac{g_0}{2\mu_0^2}\psi_0^2, \tag{1.2}$$

which may be interpreted as meaning that the φ-particle is a bound state of the ψ-particles (allocomposite particle). Similarly, in the limit $m_0^2 \to \infty$, g_0/m_0^2 finite (corresponding to $Z_2 = 0$), we obtain the constraint

$$\psi_0 = \frac{g_0}{m_0^2}\varphi_0 \psi_0, \tag{1.3}$$

meaning that the ψ-particle is composed of the φ-particle and of itself (oedipian autocomposite particle). If both renormalization constants Z_3 and Z_2 are zero, then there are no longer any elementary particles in the theory, only (gigogne) autocomposite particles.

The foregoing example is not really satisfactory, in that there is no

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classical solution to the constraint (1.3), while a particle, whether elementary or composite, can always be described in quasi-classical terms. However, examples were subsequently found [2,3] of autocomposite field models for which the compositeness constraints have a classical solution. Such examples shall be exhibited below.

Assuming then that an autocomposite field theory has a classical meaning, what is that meaning? In a classical field theory, elementary particles can be described as point singularities of the fields. If there are no elementary particles ($Z_3 = Z_2 = 0$), then there must be no singularities at the classical level; however, there may appear local concentrations of energy-momentum, which may be interpreted as extended particles. Thus, autocomposite particles, if they exist, are the same thing as localized solutions of a classical field theory, or solitons.

As will be recalled in Section 2, such solutions do indeed exist for specific autocomposite field models. The well-known properties of solitons vindicate, from the phenomenological point of view, the use of the term 'autocomposite particles' to describe these solutions. However, one would like to go a step further and set up a theory of the motion of autocomposite particles similar to that of elementary (point) particles. The purpose of the present contribution is to show how, for a specific class of autocomposite field models (in which the number of independent scalar fields is equal to the number of space dimensions), one can extract from the underlying field theory such a (relativistic) classical mechanics of autocomposite particles.

In Section 3 of this paper, it will be seen that one can associate to a multi-soliton solution of a field theory of the type considered a multiplicity of families of world-lines (one family for each soliton), thus providing a kinematical description of the multi-soliton system. Dynamics are discussed in the next sections, first from a local point of view, then from a global point of view. Local equations of motion, giving the local acceleration along each world-line in terms of effective potentials, are derived from the field equations in Section 4. The energy-momenta and the angular momenta of the various solitons of the system are obtained in Section 5 as integrals on space-like surfaces orthogonal to the corresponding families of world-lines, and the resulting global equations of motion are discussed. The case of internal symmetries of the autocomposite field theory is investigated in Section 6. In the final section, three directions for further investigation are outlined.

2. Autocomposite Scalar Fields and Solitons

Let us consider the model field theory defined by the classical Lagrangian density

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu \vec{\phi}.\partial^\mu \vec{\phi} - m_0^2 \vec{\phi}^2) + \frac{g_0}{4}(\vec{\phi}^2)^2, \tag{2.1}$$

where $\vec{\phi}$ has $(p+1)$ components. The corresponding Euler equations reduce, in the limit $m_0^2 \to \infty$, $m_0^2/g_0 = v^2$ finite, to the constraint

$$m_0^2 \vec{\phi} = g_0 \vec{\phi}(\vec{\phi}^2), \tag{2.2}$$

which means classically that the vector field $\vec{\phi}$ is constrained to vary on a p-sphere of radius v. Carrying back this constraint in the original Lagrangian, we obtain the modified Lagrangian density

$$\mathscr{L} = \tfrac{1}{2}\partial_\mu \vec{\phi}.\partial^\mu \vec{\phi} + K(\vec{\phi}^2 - v^2), \tag{2.3}$$

where K is a Lagrange multiplier. It may be recalled that Lagrangian densities such as (2.3) are closely related [3–5] to Sugawara–Sommerfeld current-algebra models [6].

From now on, we shall choose the number p of independent scalar fields equal to the number n of space dimensions. It will be seen in the next section that this choice ensures the possibility of a satisfactory kinematical description of autocomposite particles. That it allows dynamically the existence of such solutions to the field equations will be recalled now, on specific examples.

(a) $n = 1$. Putting $\phi^1 = v \cos\theta$, $\phi^2 = v \sin\theta$, the theory defined by (2.3) is simply that of a free field θ subject to a periodicity condition; this does not admit localized solutions. However we can allow, in our heuristic derivation, for the possibility of dynamical symmetry breaking by modifying the final Lagrangian density to

$$\mathscr{L} = \tfrac{1}{2}\partial_\mu \vec{\phi}.\partial^\mu \vec{\phi} + \mu^2 v \phi^1 + K(\vec{\phi}^2 - v^2). \tag{2.4}$$

This defines the well-known sine-Gordon model, for which one has analytical multi-soliton solutions. It will be convenient for our purpose to rewrite the Lagrangian density (2.4) in the 'normal' form:

$$\mathscr{L} = \tfrac{1}{2}F(\varphi)[\partial_\mu \varphi \partial^\mu \varphi - \lambda^2], \tag{2.5}$$

where

$$F(\varphi) = (\mathrm{ch}(\mu/\lambda)\varphi)^{-2}, \tag{2.6}$$

and $\lambda = 2\mu\nu$ (the usual form of the sine-Gordon model may be recovered by putting $\varphi = (\lambda/\mu)$ Log tg $\theta/4$).

(b) $n = 2$. In this case, the model defined by Lagrangian density (2.3) has analytical static k-soliton (or k-antisoliton) solutions for any k. These solutions were first obtained by Belavin and Polyakov [7] in the context of two-dimensional ferro-magnets, and were rediscovered independently by the author [8] and Matsuda [9]. Moreover, such solutions were extended to the case of fields taking their values in any surface Σ homeomorph to the sphere, for any geometry of space [8], and the resulting static general-relativistic gravitational fields were also computed [8]. These results have again recently been extended by Peremolov [10], who has shown the existence of static multi-soliton solutions for free fields in two space dimensions taking their values in any simply-connected compact homogeneous Kähler manifold.

(c) $n = 3$. A scaling argument due to Derrick [11] shows that there can be no localized solution for scalar field theories in more than two space dimensions if the Lagrangian is quadratic in the derivatives. To evade Derrick's argument, the Lagrangian density (2.3) can be modified to allow for a quartic derivative coupling. This was first done by Skyrme [12], who proposed a model given by a Lagrangian density of the type

$$\mathscr{L} = \tfrac{1}{2}\partial_\mu\vec{\phi}.\partial^\mu\vec{\phi} + \tfrac{1}{4}\gamma\,[(\partial_\mu\vec{\phi}.\partial^\mu\vec{\phi})^2 - (\partial_\mu\vec{\phi}.\partial_\nu\vec{\phi})(\partial^\mu\vec{\phi}.\partial^\nu\vec{\phi})] + K(\vec{\phi}^2 - v^2). \tag{2.7}$$

The quartic term in (2.7) can be viewed as a one-loop counterterm in the superpropagator regularization of (2.3) [13]. Skyrme has proved the existence of a static localized solution to this model, but the analytical form of this solution is not known. There are strong indications [12, 13] that such a model might provide the basis for a realistic hadrodynamics.

3. Kinematics

The heuristic line of reasoning we have followed at the beginning of the preceding section may be extended to other Lagrangian densities, such as

$$\mathscr{L} = \tfrac{1}{2}\partial_\mu\vec{\phi}.\partial^\mu\vec{\phi} - F[A(\vec{\phi})], \tag{3.1}$$

to which correspond the autocompositeness constraint

$$A(\vec{\phi}) = A_0, \tag{3.2}$$

where A_0 is an extremum of the function $F(A)$. Thus we are led to study, as autocomposite field models, theories of fields defined on $(n+1)$-dimensional curved space-time M and valued on an n-manifold Σ, which shall be assumed to be homeomorph to the n-sphere, and differentiable.

For $n = 2$ or 3, Lagrangian densities such as (2.3) or (2.7) depend only on the derivatives of the field; a trivial solution of the corresponding Euler equations is the vacuum solution which maps all points of M on an arbitrary fixed point Ω of Σ. For $n = 1$, the Lagrangian density is of the type

$$\mathscr{L} = \tfrac{1}{2}\partial_\mu\vec{\phi}.\partial^\mu\vec{\phi} - V(\vec{\phi}) + K[A(\vec{\phi}) - A_0], \tag{3.3}$$

and the vacuum solution maps all points of M on a point Ω of Σ for which the potential V is minimum. Localized solutions of the field theory are then (for any n) defined as non-constant smooth functions which map the world-line at spatial infinity ($\vec{x} = \infty$) on Ω (this definition is easily seen to be relativistically invariant). The existence of such solutions shall be assumed in this section.

Let us choose on $\Sigma - \{\Omega\}$ a local system of coordinates X^a $(a = 1, \ldots, n)$ such that $\vec{X}_\Omega = \infty$, and let the distance on this manifold be

$$dS^2 = \gamma_{ab}(\vec{X})\,dX^a\,dX^b. \tag{3.4}$$

The n-component field $\vec{\varphi}$ is then defined as the mapping $x \to \vec{X} = \vec{\varphi}(x)$. The associated topological current is

$$k^\mu = \frac{1}{n!}\frac{\sqrt{\gamma}}{\sqrt{|g|}}\varepsilon^{\mu\nu\ldots\sigma}\varepsilon_{a\ldots c}\partial_\nu\varphi^a \ldots \partial_\sigma\varphi^c, \tag{3.5}$$

where $\gamma = \det\gamma_{ab}$, $g = \det g_{\mu\nu}$ ($g_{\mu\nu}(x)$ being a suitable metric on M, with signature $+ - \ldots -$), and

$$A = \int d^n X\,\sqrt{\gamma}. \tag{3.6}$$

The current k^μ being divergenceless, the topological charge

$$K = \int d^n x\sqrt{|g|}\,k^0 = A^{-1}\int d^n x\sqrt{\gamma}\det(\partial\vec{\varphi}/\partial\vec{x}) \tag{3.7}$$

is a constant of the motion.

At any fixed time, the mapping $\vec{\varphi}$ may be locally inverted so that, to each regular point $\vec{X}$ of Σ (such that $\det(\partial\vec{\varphi}/\partial\vec{x}) \neq 0$), there correspond l

points $\vec{x}_{(i)}(\vec{X},t)$ of the n-surface $t = \text{constant}$. Thus,

$$K = \Lambda^{-1} \sum_{i=1}^{l} \int d^n X \left|\det(\partial\vec{x}/\partial\vec{\varphi})\right| \sqrt{\gamma} \det(\partial\vec{\varphi}/\partial\vec{x}) = k \tag{3.8}$$

where k is the degree of the mapping:

$$k = \sum_{i=1}^{l} \operatorname{sign} \det(\partial\vec{\varphi}/\partial\vec{x}) \tag{3.9}$$

for an arbitrary regular point $\vec{X}$. We can also write

$$l = l_+ + l_-, \qquad k = l_+ - l_-, \tag{3.10}$$

where l_+ and l_- are the number of antecedents of $\vec{X}$ with positive and negative signature respectively.

The coordinates X^a can always be chosen such that γ_{ab} is maximum and equal to δ_{ab} for $\vec{X} = 0$. In this case, the topological charge density is maximum in the vicinity of the points $\vec{x}_{(i)}(0,t)$, which may be considered as the "centres" of the various solitons. The solution $\vec{\varphi}(\vec{x},t)$ may then be interpreted as representing a system of l_+ solitons and l_- antisolitons.

Letting now t vary, to each regular point $\vec{X}$ of Σ there correspond l world-lines $\Gamma_{(i)}(\vec{X})$ of equations

$$\varphi^a(\boldsymbol{x}_{(i)}) = X^a. \tag{3.11}$$

In particular, one may view the 'zero-world-lines' $\Gamma_{(i)}(0)$ as the world-lines of a fictitious system of l point particles 'equivalent' to the real system of extended particles. In fact, this definition is somewhat arbitrary, as the $\vec{X}$-world-lines for any given $\vec{X}$ contain qualitatively the same information and may be used to define another 'equivalent' system. Moreover, a full kinematical description of the system obviously necessitates the use of all the $\vec{X}$-world-lines.

It should be noted that neither l_+, l_-, or l needs be a constant of the motion. However, the 'soliton number' k (the topological charge) being constant, the number l of $\vec{X}$-world-lines is constant modulo 2. Thus we have at the classical level the possibility of pair creation or annihilation of world-lines (as may be observed for instance in the sine-Gordon model, in the case of soliton-antisoliton scattering). In this first paper, which is mainly concerned with the similarities between classical mechanics for autocomposite particles and for point particles, it shall be assumed that these phenomena do not occur, so that the number l of world-lines is a constant of the motion. Noting that the world-lines are the lines of current for the topological current k^μ defined by (3.5), a sufficient condition for this is that the vector $\boldsymbol{k}$ is everywhere time-like. We shall assume this stronger condition.

We now proceed to set up a relativistic kinematical description in terms of co-moving coordinates. Putting

$$G^{ab} = g^{\mu\nu}\partial_\mu\varphi^a\partial_\nu\varphi^b = \boldsymbol{\partial}\varphi^a.\boldsymbol{\partial}\varphi^b, \tag{3.12}$$

it is easily seen that the condition that $\boldsymbol{k}$ is time-like is equivalent to the condition that the matrix G^{ab} is negative definite. This enables us to define, at each space-time point $\boldsymbol{x}_{(i)}$ on the world-line $\Gamma_{(i)}(\vec{X})$, n linearly independent basis vectors $\underset{(i)}{\boldsymbol{E}}_a$ orthogonal to the world-line by

$$\underset{(i)}{\boldsymbol{E}}_a = \frac{\partial \boldsymbol{x}_{(i)}}{\partial X^a} = \underset{(i)}{G}_{ab}\,\boldsymbol{\partial}\varphi^b(\boldsymbol{x}_{(i)}) \tag{3.13}$$

where G_{ab} is the matrix inverse of G^{ab}.

Let us now define the local $(n+1)$-velocity $\boldsymbol{u}_{(i)}$ along the world-line $\Gamma_{(i)}(\vec{X})$ by

$$u^\mu = \frac{k^\mu}{\|k\|} = \frac{1}{n!}\frac{\sqrt{|\bar{G}|}}{\sqrt{|g|}}\varepsilon^{\mu\nu\ldots\sigma}\varepsilon_{a\ldots c}\,\partial_\nu\varphi^a \ldots \partial_\sigma\varphi^c, \tag{3.14}$$

where $\bar{G} = \det G_{ab}$, and the local proper time along the world-line by

$$ds_\Gamma = \boldsymbol{u}.d\boldsymbol{x}, \tag{3.15}$$

from which it also follows that

$$\boldsymbol{u} = \frac{\partial \boldsymbol{x}}{\partial s_\Gamma} \tag{3.16}$$

(derivative along the world-line). The time-like coordinate X^0 is then defined by

$$dX^0 = \sqrt{G^{00}}\,ds_\Gamma, \tag{3.17}$$

where $G^{00}(\boldsymbol{x})$ is such that dX^0 is a perfect differential[1] (note that, due to the arbitrariness in the definition of X^0, it is always possible to choose this coordinate so that, for instance, $G^{00}(X^0, \vec{X} = 0) = 1$). The corresponding basis vector

$$\boldsymbol{E}_0 = \sqrt{G_{00}}\,\boldsymbol{u}, \tag{3.18}$$

where $G_{00} = (G^{00})^{-1}$, is by construction orthogonal to the n basis vectors $\boldsymbol{E}_a$.

We have thus defined a system of curvilinear coordinates X^α ($\alpha = 0, 1, \ldots, n$), and the associated metric $G_{\alpha\beta}$ (with $G_{0a} = 0$). The components of the $(n+1)$-velocity in the E-basis are

$$U^\alpha = \frac{\partial X^\alpha}{\partial s_\Gamma} = \sqrt{G^{00}}\,\delta_0^\alpha \tag{3.19}$$

which means that the X-coordinates are co-moving.

In the case of a non-static solution, it is interesting to compute the covariant derivatives of the local velocity $\boldsymbol{u}_{(i)}$. These are given by

$$D_\beta \boldsymbol{u} = \nabla_\beta U^\alpha \boldsymbol{E}_\alpha = \Gamma^a_{\beta 0} U^0 \boldsymbol{E}_a \tag{3.20}$$

in co-moving coordinates, where $D_\beta = \partial/\partial X^\beta$, and $\Gamma^\alpha_{\beta\gamma}$ is the X-connexion. Thus the local covariant acceleration, which measures the deviation of the world-lines from geodesics, is given in X-coordinates by

$$A^a = \nabla U^a/\nabla s_\Gamma = -G^{ab} D_b \operatorname{Log}\sqrt{G_{00}}, \tag{3.21}$$

and the local velocity dispersion is, in X-coordinates,

$$\nabla_b U^a = \frac{1}{2} G^{ac} \frac{\partial G_{bc}}{\partial s_\Gamma}. \tag{3.22}$$

4. Local equations of motion

The dynamics of a theory of a p-component field $\vec{\varphi}(x)$ may be derived from an action principle, with the action

$$S = \int \mathrm{d}^{n+1} x \sqrt{|g|}\, \mathscr{L}(\vec{\varphi}, \partial_\mu \vec{\varphi}). \tag{4.1}$$

Relativistic invariance of the theory demands that the Lagrangian density $\mathscr{L}$ depends on the field derivatives only through the scalar products $\partial\varphi^a . \partial\varphi^b = G^{ab}$. Specializing now to an autocomposite field theory, with $p = n$, and carrying out the coordinate transformation $x^\mu \to X^\alpha$ introduced in the preceding section, the action may be written as a sum over the families of world-lines:

$$S = \sum_{i=1}^{l} \int \mathrm{d}^{n+1} X \sqrt{|\underset{(i)}{G}(\vec{X})|}\, \mathscr{L}(\vec{X}, \underset{(i)}{G}{}^{ab}(\vec{X})), \tag{4.2}$$

where $G = \det G_{\alpha\beta} = G_{00}\bar{G}$. All that follows shall be derived from this form of the action.

Varying the action (4.2) with respect to the unknown 'fields' $G_{\alpha\beta}$ yields

$$\delta S = -\frac{1}{2} \sum_{i=1}^{l} \int \mathrm{d}^{n+1} X \sqrt{|\underset{(i)}{G}|}\, \underset{(i)}{T}{}^{\alpha\beta}\, \delta \underset{(i)}{G}{}_{\alpha\beta}, \tag{4.3}$$

where $T^{\alpha\beta}$ is the canonical energy-momentum tensor in X-coordinates:

$$T^{\alpha\beta} = -\frac{2}{\sqrt{|G|}} \frac{\partial(\sqrt{|G|}\mathscr{L})}{\partial G_{\alpha\beta}}. \tag{4.4}$$

From

$$G_{\alpha\beta} = g_{\mu\nu} D_\alpha x^\mu D_\beta x^\nu, \tag{4.5}$$

it follows that

$$\delta G_{\alpha\beta} = 2 g_{\mu\nu} D_\alpha x^\mu D_\beta \delta x^\nu + \partial_\lambda g_{\mu\nu} D_\alpha x^\mu D_\beta x^\nu \delta x^\lambda. \tag{4.6}$$

Carrying (4.6) into (4.3), and integrating by parts, we obtain

$$\begin{aligned} \delta S = &-\sum_{i=1}^{l} \int \mathrm{d}^{n+1} X D_\beta [\sqrt{|\underset{(i)}{G}|}\, \underset{(i)}{T}{}^{\alpha\beta} g_{\mu\nu}(\boldsymbol{x}_{(i)}) D_\alpha x^\mu_{(i)} \delta x^\nu_{(i)}] \\ &+ \sum_{i=1}^{l} \int \mathrm{d}^{n+1} X \{ D_\beta [\sqrt{|\underset{(i)}{G}|}\, \underset{(i)}{T}{}^{\alpha\beta} g_{\mu\nu}(\boldsymbol{x}_{(i)}) D_\alpha x^\mu_{(i)}] \\ &- \tfrac{1}{2} \sqrt{|\underset{(i)}{G}|}\, \underset{(i)}{T}{}^{\alpha\beta} \partial_\nu g_{\lambda\mu}(\boldsymbol{x}_{(i)}) D_\alpha x^\mu_{(i)} D_\beta x^\lambda_{(i)} \} \delta x^\nu_{(i)}. \end{aligned} \tag{4.7}$$

The action principle applied to variations of the independent 'fields' $x^\mu_{(i)}(X)$ constrained to be zero on a prescribed boundary n-surface thus yields the Euler equations

$$D_\beta[\sqrt{|G|}\, T^{\alpha\beta} g_{\mu\nu} D_\alpha x^\mu] - \tfrac{1}{2}\sqrt{|G|}\, T^{\alpha\beta} \partial_\nu g_{\lambda\mu} D_\alpha x^\mu D_\beta x^\lambda = 0. \tag{4.8}$$

Because of the general covariance of the theory, this may be locally evaluated in the special coordinate system $x^\mu = X^\mu$, and we obtain (lowering indices with the metric $G_{\alpha\beta}$)

$$D_\beta[\sqrt{|G|}\, T^\beta_\alpha] - \tfrac{1}{2}\sqrt{|G|}\, T^{\beta\gamma} D_\alpha G_{\beta\gamma} = 0, \tag{4.9}$$

which means that the covariant divergence [14] of the energy-momentum tensor is (not surprisingly) zero:

$$\nabla_\beta T^\beta_\alpha = 0. \tag{4.10}$$

In fact, these $(n+1)$ equations cannot be independent, as there are only n independent fields. It follows from $G_{0a} = 0$ and from the fact that

$\mathscr{L}$ does not depend on G_{00} that

$$T^{0\beta} = -\mathscr{L} G^{0\beta}; \tag{4.11}$$

hence,

$$D_\beta(\sqrt{|G|}\,T^\beta_0) = -D_0(\mathscr{L}\sqrt{|G|}) = \tfrac{1}{2}\sqrt{|G|}\,T^{\alpha\beta} D_0 G_{\alpha\beta}, \tag{4.12}$$

using the definition (4.4) of the energy-momentum tensor. Thus Equation (4.10) for $\alpha = 0$ reduces to an identity. The n independent dynamical equations are

$$\nabla_\beta T^\beta_a = \bar{\nabla}_b T^b_a + \tfrac{1}{2} G^{00} D_b G_{00}(T^b_a - \delta^b_a T^0_0) = 0, \tag{4.13}$$

where the symbol $\bar{\nabla}_b$ stands for the spatial covariant derivative. Putting

$$T^{ab} = \tau^{ab} - \mathscr{L} G^{ab}, \tag{4.14}$$

with $\tau^{ab} = -2\partial\mathscr{L}/\partial G_{ab}$, and using (3.21), Equations (4.13) may be written

$$\bar{\nabla}_b T^b_a = \tau_{ab} A^b. \tag{4.15}$$

These equations can be inverted if $\tau = \det \tau_{ab} \neq 0$, giving the local acceleration in X-coordinates:

$$A^a = \hat{\tau}^{ac} \bar{\nabla}_b T^b_c, \tag{4.16}$$

where $\hat{\tau}_{ac}$ is the matrix inverse of τ_{ac}.

Let us discuss, as an example, the autocomposite field models for $n = 1$ or 2 defined by the Lagrangian density [8]

$$\mathscr{L} = \tfrac{1}{2}\gamma_{ab}(\vec{\varphi}) G^{ab} - V(\vec{\varphi}), \tag{4.17}$$

where γ_{ab} is the metric on $\Sigma - \{\Omega\}$ defined in Section 3, and $V = (\lambda^2/2)\gamma_{11}$ for the case $n = 1$ ('normal' form of the Lagrangian), $V = 0$ for $n = 2$. Then,

$$\tau_{ab} = \gamma_{ab}, \tag{4.18}$$

and Equations (4.16) take the form

$$A^a = G^{bc}(\hat{\Gamma}^a_{bc} - \Gamma^a_{bc}) + \hat{\gamma}^{ab} D_b V, \tag{4.19}$$

where $\hat{\Gamma}^a_{bc}$ is the connexion on $\Sigma - \{\Omega\}$.

The corresponding static equations, obtained by putting A^a equal to zero, yield the static solutions of these models [8]. In fact, these solutions were obtained in Reference 8 as solutions of the equations $T_{ab} = 0$ which

are indeed solutions of (4.15). The static conditions $T_{ab} = 0$, or the equivalent conditions in x-coordinates $T_{ij} = 0$, have also been shown generally [15] to derive from the metric-independent saturation of the topological lower bound to the action.

We are here interested in the dynamical content of equations (4.19), and more specially in the motion of the 'centres' $\vec{X} = 0$ for which $\gamma_{ab}(0) = \delta_{ab}$, $\hat{\Gamma}^a_{bc}(0) = 0$. The equations of motion of the centres are thus simply

$$A^a = -G^{bc}\Gamma^a_{bc} = \frac{1}{\sqrt{|\bar{G}|}} D_b(\sqrt{|\bar{G}|}\,G^{ab}), \qquad (\vec{X} = 0). \tag{4.20}$$

In the case $n = 1$, assuming now that space-time is flat, these can be written in x-coordinates as

$$\frac{\partial^2 x^\mu}{\partial s_\Gamma^2} = -\frac{\partial x^\mu}{\partial X}\frac{\partial x^\nu}{\partial X}\sqrt{|G^{11}|}\,\partial_\nu(\sqrt{|G^{11}|}), \qquad (X = 0), \tag{4.21}$$

which may be approximated, in the non-relativistic limit, by

$$\frac{\partial^2 x}{\partial t^2} \simeq -\left(\frac{\partial x}{\partial X}\right)^2 \sqrt{|G^{11}|}\,\partial_1(\sqrt{|G^{11}|}), \qquad (X = 0), \tag{4.22}$$

or, noting that in the non-relativistic limit $\partial x/\partial X = \pm\sqrt{|G_{11}|}$,

$$\frac{\partial^2 x_{(i)}}{\partial t^2} \simeq -\frac{\partial}{\partial x}\,\mathrm{Log}\,(\lambda^{-1}\sqrt{|G^{11}_{(i)}|}), \qquad (X = 0), \tag{4.23}$$

which are the Newtonian equations of motion for the centres.

These may be checked on the example of the sine-Gordon model, for the soliton-soliton solution in the centre-of-mass referential:

$$\varphi = \frac{\lambda}{\mu}\varepsilon(x)\,\mathrm{Log}\left|\frac{u\,\mathrm{sh}\,\mu\gamma x}{\mathrm{ch}\,\mu\gamma u t}\right|, \tag{4.24}$$

where $\gamma = (1 - u^2)^{-1/2}$. In this case we find

$$G^{11} = -\lambda^2\gamma^2(\coth^2\mu\gamma x - u^2\,\mathrm{th}^2\mu\gamma u t), \tag{4.25}$$

which gives, in the non-relativistic limit ($u \ll 1$, whence $\mu\gamma x \gg 1$ for finite φ):

$$\mathrm{Log}\,(\lambda^{-1}\sqrt{|G^{11}|}) \simeq 2\mathrm{e}^{-2\mu\gamma|x|} + \frac{u^2}{2\,\mathrm{ch}^2\mu\gamma u t} + \ldots, \tag{4.26}$$

so that the effective Newtonian potential is

$$V_{\mathrm{eff}}(x) = 2M\mathrm{e}^{-2\mu\gamma|x|}, \tag{4.27}$$

where $M = 2\lambda^2/\mu$ is the soliton mass (such a potential has also been derived by Vinciarelli [16], using a different method). It is easily seen that this potential leads approximately to the correct motion for the centres of the two solitons.

In the case $n = 2$, Equations (4.20) or their non-relativistic counterpart are of a non-Newtonian form. This is not surprising, as the existence of static n-soliton solutions means that the effective static potential is zero (the same happens for a system of gravitating point particles in two space dimensions or, in three space dimensions, for a system of charged particles whose static gravitational attraction and electric repulsion balance exactly). These equations shall be further investigated elsewhere.

5. Energy-momentum and angular momentum

It is assumed in this section that space-time M is flat, and that the metric $g_{\mu\nu}$ is Minkowskian. The total energy-momentum of the field is defined, in terms of the divergenceless energy-momentum tensor ($t^{\mu\nu}$ in x-coordinates) by

$$P^\mu = \int d\pi_\nu t^{\mu\nu}, \tag{5.1}$$

which does not depend on the space-like n-surface Π.

For a multi-soliton solution, P^μ may be written as the sum of the energy-momenta of the various solitons. There are different ways to do this depending on the choice of Π. A convenient choice is the n-surface $X^0 = \text{constant}$, for which $d\pi_\nu = d^n X \sqrt{|\bar{G}|}\, u_\nu$, so that

$$P^\mu = \sum_{i=1}^{l} P^\mu_{(i)}, \tag{5.2}$$

with

$$P^\mu_{(i)}(X^0) = \int d^n X \sqrt{|\underset{(i)}{G}|}\, u_{(i)\nu} t^{\mu\nu}_{(i)}. \tag{5.3}$$

Carrying out the coordinate transformation $X^\alpha \to x^\mu$, and using (3.13), (3.16), and (3.17), we obtain

$$t_{\mu\nu} = T^0_0 u_\mu u_\nu + T_{ab}\partial_\mu \varphi^a \partial_\nu \varphi^b. \tag{5.4}$$

Hence, taking into account (4.11) and the orthogonality of $\boldsymbol{u}$ and $\partial\varphi^a$,

$$P^\mu_{(i)}(X^0) = \int d^n X \sqrt{|\underset{(i)}{G}|}\,(-\mathscr{L}_{(i)}) u^\mu_{(i)}. \tag{5.5}$$

This formula for the energy-momentum of the ith soliton may alternatively be derived directly from the action principle. If the action is computed between the two boundary n-surfaces $X^0(1)$ and $X^0(2)$, the variation of the action resulting from an infinitesimal variation of the n-surface $X^0(2)$ is, according to (4.7),

$$\begin{aligned} \delta S &= -\sum_{i=1}^{l} \int d^{n+1} X D_\beta [\sqrt{|\underset{(i)}{G}|}\, T^{\alpha\beta}_{(i)} g_{\mu\nu} D_\alpha x^\mu_{(i)} \delta x^\nu_{(i)}] \\ &= -\sum_{i=1}^{l} \int_{X^0(2)} d^n X \sqrt{|\underset{(i)}{G}|}\, T^0_{(i)0} u_{(i)\nu} \delta x^\nu_{(i)}, \end{aligned} \tag{5.6}$$

using Gauss's theorem. The variational definition of the energy-momentum

$$P_{(i)\mu} = -\frac{\delta S}{\delta x^\mu_{(i)}}, \tag{5.7}$$

then yields again formula (5.5).

In the case of a scattering solution, we expect that, when $X^0 \to \pm\infty$, the local velocities tend to constants. It follows from the local equations of motion (4.16) that the metric coefficients G_{ab} tend to the static solution, and Equations (3.22) then imply that there can be no dispersion. Asymptotically, the energy-momentum of the ith soliton is therefore

$$P^\mu_{(i)}(\infty) = M u^\mu_{(i)}(\infty), \tag{5.8}$$

where M is the soliton mass (in the case $n = 2$, although there are an infinity of static solutions, the static energy per soliton is still unique [8]).

The total energy-momentum (5.1) is a constant of the motion, but not the individual soliton energy-momenta (except for a static solution). To compute their dependence on X^0 we use the Euler equations (4.8) which, for a Minkowskian metric $g_{\mu\nu}$, reduce to

$$D_0(\sqrt{|G|}(-\mathscr{L})G^{00} D_0 x^\mu) + D_a(\sqrt{|G|}\, T^{ab} D_b x^\mu) = 0. \tag{5.9}$$

It follows that

$$D_0 P^\mu_{(i)} = -\int d^n X D_a(\sqrt{|\underset{(i)}{G}|}\, T^{ab}_{(i)} D_b x^\mu_{(i)}), \tag{5.10}$$

which gives, using Gauss's theorem,

$$D_0 P^\mu_{(i)} = -\int_{\sigma_\infty(X^0)} d\sigma_a \sqrt{|\underset{(i)}{G}|}\, T^a_{(i)b} \partial^\mu \varphi^b_{(i)}, \tag{5.11}$$

where $\sigma_\infty(X^0)$ is an $(n-1)$-surface at $\vec{X}$-infinity in the n-surface X^0.

Equations (5.11) are the global equations of motion for the various solitons. It is remarkable that the 'forces' (the right-hand sides of these equations) depend only on the behaviour of the fields at $\vec{X}$-infinity, that is in the vicinity of the point Ω of the compact manifold Σ. Note that the convergence of the integrals (5.5) implies $\lim_{\vec{X}\to\infty} |\vec{X}|^n \sqrt{|\bar{G}|}\,\mathscr{L} = 0$. We shall assume that this implies also $\lim_{\vec{X}\to\infty} |\vec{X}|^n \sqrt{|\bar{G}|}\, T^b_a = 0$. and that G_{00} stays finite everywhere. Then a necessary condition for the existence of non-static multi-soliton solutions is that, at least on portions of $\sigma_\infty(X^0)$, $\partial_\mu \varphi^b \to \infty$ (more quickly than $|\vec{\varphi}|$), which implies $G_{ab} = 0$.

In the particular case $n = 1$, the point Ω has, at fixed X^0, l antecedents, one of which is $x = \infty$, and the $(l-1)$ others at finite distance 'separate' the various solitons. For a scattering solution, the field φ in the vicinity of the outermost solitons may be approximated, at sufficiently large times, by Lorentz-transforms of the static solution $\varphi(x) = \pm\lambda(x-a)$ [8], so that $\partial_\mu\varphi \to$ constant; thus only the 'separators' contribute to the global acceleration (5.11).

For instance, for soliton-soliton scattering, the separator is, by reason of symmetry, the centre of mass x_G. In the centre-of-mass referential ($x_G = 0$) the energies of the two solitons are obviously equal and constant, while their momenta $P^1_\pm$ obey

$$D_0 P^1_\pm = \mp\sqrt{G_{00}}\, T^1_1]_{x=0} = \pm\tfrac{1}{2}\sqrt{G_{00}}\, F|G^{11}|]_{x=0}. \tag{5.12}$$

Nothing that, along the centre-of-mass world-line Γ_G, $\mathrm{d}X^0 = \sqrt{G^{00}}\,\mathrm{d}t$, this may be integrated to give

$$P^1_\pm = \pm\left[\int_0^t \mathrm{d}t\,\tfrac{1}{2} F|G^{11}|\right]_{x=0}. \tag{5.13}$$

In the example of the sine-Gordon model, this procedure yields

$$P^1_\pm = \pm M\gamma u \operatorname{th} \mu\gamma u t \tag{5.14}$$

(which may also be obtained by direct integration of the momentum density t^{01} on the half-line t = constant, $x > 0$), where t may be re-expressed in terms of X^0 (measured along Γ_G). This may be compared with the purely kinetic contribution

$$Mu^1 = \pm M\lambda|G^{11}|^{-1/2} u \operatorname{th} \mu\gamma u t, \tag{5.15}$$

along the 0-world-line, with G^{11} given by (4.25).

The method used to define the energy-momenta of the various solitons may also be applied to the angular momentum tensor. The total angular momentum relative to the origin of x-coordinates is

$$M^{\mu\nu} = \int \mathrm{d}\pi_\lambda (t^{\mu\lambda}x^\nu - t^{\nu\lambda}x^\mu). \tag{5.16}$$

Choosing for Π the n-surface X^0 = constant, this may be written as

$$M^{\mu\nu} = \sum_{i=1}^{l} M^{\mu\nu}_{(i)}, \tag{5.17}$$

with the angular momentum of the ith soliton given by

$$M^{\mu\nu}_{(i)}(X^0) = \int \mathrm{d}^n X \sqrt{|\underset{(i)}{\bar{G}}|}(-\mathscr{L}_{(i)})(u^\mu_{(i)} x^\nu_{(i)} - u^\nu_{(i)} x^\mu_{(i)}), \tag{5.18}$$

which may also be derived from the action principle via (5.6). The method already used for the energy-momentum gives for the X^0-derivative of this angular momentum:

$$D_0 M^{\mu\nu}_{(i)} = -\int_{\sigma(X^0)} \mathrm{d}\sigma_a \sqrt{|\underset{(i)}{G}|}\, T^{ab}_{(i)} (D_b x^\mu_{(i)} . x^\nu_{(i)} - D_b x^\nu_{(i)} . x^\mu_{(i)}). \tag{5.19}$$

6. Internal symmetries

For the autocomposite field theories under study, an internal symmetry may be defined as a local coordinate transformation on $\Sigma - \{\Omega\}$ which leaves the action (4.1) or (4.2) invariant. For the $n = 2$ models, as the Lagrangian density (4.17) with $V = 0$ depends only on the invariant $\gamma_{ab} G^{ab}$, these transformations are the general coordinate transformations. The same is true of the Skyrme model defined by (2.7).

For any infinitesimal symmetry transformation $\varphi^a \to \varphi^a + \psi^a(\vec{\varphi})$, the current

$$J^\mu = \frac{\partial\mathscr{L}}{\partial(\partial_\mu\varphi^a)}\psi^a = \tau_{ab}\partial^\mu\varphi^b\psi^a \tag{6.1}$$

is divergenceless:

$$\partial_\mu J^\mu = 0. \tag{6.2}$$

It follows that the charge defined by

$$Q = \int \mathrm{d}\pi_\nu J^\nu \tag{6.3}$$

does not depend on the space-like n-surface Π. We may take advantage of this fact to compute the charge Q on the surface $X^0 = $ constant, and obtain the very simple result

$$Q = \sum_{i=1}^{l} \int \mathrm{d}^n X \sqrt{|\underset{(i)}{\bar{G}}|}\, u_{(i)\nu} J^\nu_{(i)} = 0, \tag{6.4}$$

owing to the orthogonality of $\boldsymbol{u}$ and $\partial\varphi^a$. This result may again be derived directly from the action principle using (5.6), and noting that $T^{0a} = 0$.

Thus, the dynamical charge carried by any localized solution of an autocomposite field theory is identically zero, provided (3.17) is integrable.[1] While this does not completely exclude the possibility of localized solutions with non-zero charges, it is doubtful whether these exist for $n = 2$ (in this case, the one-soliton time-dependent charged solution found by Duff and Isham [17] can be shown to be non-localizable.

7. Conclusion

In this paper, we have investigated a formulation of classical mechanics for autocomposite particles, in a class of models of n interacting scalar fields on $(n+1)$-dimensional space-time. The key to this formulation is that a localized solution of these models is a mapping (of space-time into an n-manifold) of degree k. This mapping may be inverted, so that the original fields may be taken as local space-like coordinates X^a, a local time-like coordinate X^0 being then defined so that the coordinates X^a are co-moving. The l inverse mappings ($l \geqslant |k|$; we have restricted ourselves to the case $l = $ constant) $\boldsymbol{x}_{(i)}(X)$ give the original (e.g. Minkowskian) coordinates as a set of l fields depending on the co-moving coordinates X^a. The classical mechanics of the l solitons may then be extracted from the classical theory of these fields.

After this preliminary exploration, further investigations should be conducted along the following three major axes:

(1) *Classical mechanics.* Among the many problems which remain are:
– the extension of the present formalism to the case of non-constant l (implying classical, real or virtual, pair creation or annihilation);
– the definition of the centre of mass of a soliton, which is obvious only in the static case;
– the study of the scattering of autocomposite particles;
– the Hamiltonian formulation;
– the finding of exact or approximate solutions, for the cases $n = 2$ or 3, of Equations (4.16), supplemented by the condition that space-time is flat, i.e. that the X-Riemann tensor vanishes:

$$R_{\alpha\beta\gamma\delta} = 0, \tag{7.1}$$

which for n = 2 reduces to the vanishing of the X-Ricci tensor,

$$R_{\alpha\beta} = 0; \tag{7.2}$$

for the case $n = 2$, a possible tool for this might be a suitable modification of the post-Newtonian approximation (used in general relativity), starting from the known static solutions.

(2) *General relativity.* The coupled Einstein-autocomposite field equations have been solved for the static $n = 2$ case in [8]. The present formalism being generally covariant is appropriate for the treatment of the dynamical case. The dynamical equations are then the Einstein equations in X-coordinates

$$R_{\alpha\beta} - \tfrac{1}{2} R G_{\alpha\beta} = \chi T_{\alpha\beta}, \tag{7.3}$$

from which follow Equations (4.10), hence also Equations (4.16). The problems which should then be investigated are:
– the finding of (approximate) solutions to these equations;
– the general-relativistic definition of total energy-momentum and angular momentum of autocomposite particles;
– in the case $n = 3$, the possibility of solutions with unusual space-time topology.

(3) *Quantum mechanics.* It has been suggested [16] that the knowledge of the classical mechanical properties of solitons might be useful for their approximate quantization, via the correspondence principle. A more radical approach, in the spirit of the classical formalism presented here, would be to quantize the inverse 'fields' $x^\mu_{(i)}(X)$ using the canonical quantization procedure (field quantization in curvilinear coordinates on flat space-time), and to extract (just as in the classical case) the quantum mechanics of the solitons from the quantum theory of their coordinate fields. This should provide an interesting alternative to the usual soliton quantization procedure [18] based on collective coordinates. For the

case $n = 3$, this approach should be used to investigate the generation of spin [19], which is known to appear, at the quantum level, in Skyrme's model [12, 20], and of isospin.

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Note

[1] dX^0 is a perfect differential if the integrability conditions $\varepsilon^{\mu\nu\rho} u_\mu \partial_\nu u_\rho = 0$ are satisfied; these are of course identically satisfied for $n = 1$, but not necessarily for $n \geq 2$. However, most of the following results may readily be extended to the case of an arbitrary time-like coordinate X^0.

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EXCLUSION OF STATIC SOLUTIONS IN GRAVITY-MATTER COUPLING

ABSTRACT. Coupled Einstein-matter systems of various spins are shown to possess no static non-singular solutions. In particular, this is demonstrated for spin 1/2 with or without torsion by iterating the Dirac operator, and partial results are given for supergravity.

Introduction

It is well-known that there are no non-singular static solutions for free massless or massive fields of any spin in flat space, basically as a consequence of the energy-momentum relations governing propagation of their excitations. This property persists also (at least in four dimensions) in non-Abelian gauge theories, including general relativity, despite their self-coupling. There is therefore no reason to expect dramatic changes when the matter systems are coupled to gravity. However, the derivation is sufficiently different in the case of spinor fields to warrant discussion. Complications arise here for two reasons: in contrast to spin 0 or 1, covariant derivatives are unavoidable and the necessity of dealing with the second-order form of the Dirac equation brings in curvature dependence explicitly. These difficulties are also of interest as an introduction to the same problem in supergravity, representing a new type of gauge system.

After reviewing pure gravity and its coupling to spins 0 and 1, we will consider spin 1/2 and discuss the associated second-order equation it satisfies in presence of gravity, with or without torsion. The desired result will then be established there. We conclude with some partial results for supergravity based on the iterated Rarita–Schwinger equation.

Gravity-integral spin coupling

Static gravitational fields are characterized by the existence of coordinate systems in which both all time dependence ($\partial_0 g_{\mu\nu}$) and the mixed (g_{0i}) components of the metric vanish. Although there are no non-singular solutions in the more general stationary ($g_{0i} \neq 0$) situation [1] either, we shall restrict ourselves to the static case throughout. The Einstein equations in the presence of sources reduce to

$$ {}^3G^{ij} + N^{-1}(g^{ij}\nabla^2 - D^iD^j)N = \tfrac{1}{2}T^{ij} \tag{1} $$

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